JOURNAL OF FUNCTIONAL ANALYSIS 66, 283–307 (1986)

Belated Integrals

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Communicated by L. Gross

Received October 29, 1984

INTRODUCTION

Recent theories of the Non-commutative Stochastic Integral have indicated that it is a kind of vector Riemann–Stieltjes integral. In [6] Bartle develops a bilinear vector integral which at first sight appears to be a natural setting for the non-commutative Stochastic Integral. However, in [2, remarks following 7: 18] it was shown that the Non-commutative Stochastic Integral could *not* be obtained using Bartle's vector integral. In this paper we show that by weakening a few of the requirements of Bartle's theory (on $\mathbb{R}^+$) and abstracting some of the structure employed in [2], we get an integration theory on $\mathbb{R}^+$ which has as special cases each of the integrals described in [2–5], and reduces to Bartle's integral in a special case, thus unifying these distinct theories.

1. NOTATIONS

We shall set up our integration theory using a particular field $\mathscr{F}$ of subsets of $\mathbb{R}^+$. The use of this field is not essential. What is required is a field consisting of sets which are finite unions of intervals, usually of some given type, e.g., left open and right closed. So let $\mathscr{F}$ be the field of subsets of $[0, T]$, $T > 0$, comprising finite unions of subintervals of $[0, T]$. The intervals may be with or without a particular endpoint. Let μ be a finitely additive set function on $\mathscr{F}$ with values in a linear space Y. Let X be a nor-

0022-1236/86 $3.00

med space and (X_α), $\alpha \in [0, T]$, an increasing family of subspaces of X (see 7.1 of [2]). We suppose that there is a bilinear multiplication $X \times Y \to Z$, where Z is a Banach space. Bartle develops his theory using the semivariation of μ with respect to X, which for $E \in \mathscr{F}$ is defined as

$$\|E\| = \sup \left| \sum_i x_i \mu E_i \right|, \tag{1.01}$$

where the supremum is taken over all partitions of E in $\mathscr{F}$ into a finite number of disjoint sets E_i and finite collections $\{x_i\}$ with $|x_i| \leqslant 1$. We shall work with the right-belated semivariation of μ with respect to (X_α). For $E \in \mathscr{F}$ this is defined as

$$\|E\|^b = \sup \left| \sum_i x_i \mu E_i \right| \tag{1.02}$$

but now the supremum is taken over all partitions of E into a finite number of disjoint intervals E_i, in $\mathscr{F}$, and all collections $\{x_i\}$ with $\forall i\, x_i \in X_{\inf E_i}$ and $|x_i| \leqslant 1$. It may at times be necessary to emphasise the dependence of $\|\cdot\|^b$ upon μ. In this case we will write $\|\cdot\|^b_\mu$ rather than $\|\cdot\|^b$.

Remarks. (i) We restrict our attention to $[0, T]$ rather than $\mathbb{R}^+$ to avoid technical difficulties associated with infinite measure spaces.

(ii) In [6] the linear space Y is normed and multiplication satisfies $|xy| \leqslant k|x|\,|y|$, $k > 0$. We have not found these assumption necessary for the general theory. However, these conditions are realised and utilised in some of the applications of our theory (see Section 4).

(iii) The bilinear multiplication $X \times Y \to Z$ is represented above as xy. The order in which we write x and y in the product is of course quite arbitrary: xy just means the product of x and y. It may be that in a specific situation xy and yx make (possibly quite different) sense. The formalism for products we have adopted can deal with both cases at once. However, we feel that it will make matters clearer if a distinction between xy and yx is maintained. Thus if there is a natural bilinear multiplication $Y \times X \to Z'$, where Z' is a Banach space (possibly different from Z), we would then work with the left-belated semivariation of μ with respect to (X_α), defined by, $E \in \mathscr{F}$,

$$^b\|E\| = \sup \left| \sum \mu\, E_i x_i \right|. \tag{1.03}$$

The supremum is taken as in (1.02). We shall develop the theory using the right-belated semivariation leaving the reader to work through the "left-belated" results.

In the next few sections we discuss measurability and integrability with respect to $\|\cdot\|^b$. Essentially all that we do is to reformulate Section 1, 2, and 3 of [6] using the belated semivariation instead of the semivariation.

Proofs are sketched so that the reader may see how the belated semivariation takes over the role of the semivariation. Since we shall refer to $[0, T]$ very often we shall write $\mathbf{T}$ for $[0, T]$.

2. Simple Processes and Measurability

First we develop some of the properties of the right-belated semivariation.

2.1. Lemma. *Let $E \in \mathscr{F}$; then*

(i) $\|E\|^b \leqslant \|E\|$,

(ii) $\|\cdot\|^b$ *is monotone and subadditive on* $\mathscr{F}$.

Proof. (i) Let $\{E_i\} \subset \mathscr{F}$ be a partition of E and $\{x_i\} \subset X$ be such that $\forall i\ |x_i| \leqslant 1$ and $x_i \in X_{\inf E_i}$. Then by definition of $\|E\|$, $|\sum_i x_i \mu E_i| \leqslant \|E\|$. Taking the *appropriate* supremum on the left-hand side confirms the result.

(ii) Let $E, F \in \mathscr{F}$ and suppose that $E \cap F = \varnothing$. Let $\{G_i\}$ be a partition of $E \cup F$ into a finite number of disjoint intervals. Let $\{x_i\}$ be such that $\forall i\ x_i \in X_{\inf G_i}$ and $|x_i| \leqslant 1$ and set $E_i = E \cap G_i$, $F_i = F \cap G_i$. Now for a fixed i, $E_i = \bigcup_j E_{ij}$, where $\{E_{ij}\}$ is a finite number of disjoint intervals. We note that $x_i \in X_{\inf E_{ij}}$ for each j. Similarly $F_i = \bigcup_i F_{ij}$ and $x_i \in X_{\inf F_{ij}}$ for each j. Now $\{E_{ij}\}$ is a partition of E into intervals and $\{F_{ij}\}$ a similar partition of F. Hence

$$\begin{aligned}\left|\sum_i x_i \mu G_i\right| &= \left|\sum_i x_i \mu(G_i \cap (E \cup F))\right| \\ &= \left|\sum_i \sum_j x_i \mu E_{ij} + \sum_i \sum_j x_i \mu F_{ij}\right| \\ &\leqslant \left|\sum_i \sum_j x_i \mu E_{ij}\right| + \left|\sum_i \sum_j x_i \mu F_{ij}\right| \\ &\leqslant \|E\|^b + \|F\|^b.\end{aligned}$$

Thus $\|E \cup F\|^b \leqslant \|E\|^b + \|F\|^b$. To prove the monotonicity take $E, F \in \mathscr{F}$ and suppose that $E \subseteq F$. Let $\{E_i\}$ be a partition of E into intervals and $\{x_i\}$ such that $\forall i\ x_i \in X_{\inf E_i}$ and $|x_i| \leqslant 1$. Let $\{G_i\}$ be a partition of $F \backslash E$ into intervals. Then

$$\left|\sum x_i \mu E_i\right| = \left|\sum_i x_i \mu E_i + O \cdot \mu G_i\right| \leqslant \|F\|^b.$$

Taking the appropriate supremum on the left-hand side gives the result.

Q.E.D.

We can extend $\|\cdot\|^b$ to all subsets of $\mathbf{T}$.

2.2. DEFINITION. Let $A \subseteq \mathbf{T}$. We define $\|A\|^b$ by $\|A\|^b = \inf\{\|E\|^b$: $E \in \mathscr{F}\ E \supseteq A\}$. It is straightforward to show that $\|\cdot\|^b$ extended in this way is monotone and subadditive. We shall call A μ^b-null ("μ-flat null") if $\|A\|^b = 0$. A property $P(s)$ will be said to hold μ^b-almost everywhere if $\|\{s$: not $P(s)\}\|^b = 0$.

Remark. We shall see later that an alternative definition of $\|A\|^b$ for arbitrary $A \subseteq \mathbf{T}$ is often desirable; cf. 2.11 onwards.

2.3. DEFINITION. (i) A map $f\colon \mathbf{T} \to X$ is a process if $f(s) \in \bar{X}_s$, $s \in \mathbf{T}$, where $\bar{X}_s$ is the closure of X_s in X.

(ii) A process is elementary if it has the form $x\chi_I(s)$, where I is an interval and $\|I\|^b < \infty$ and $x \in X_{\inf I}$.

(iii) A process is simple if it is a linear combination of elementary processes with disjoint supports.

(iv) For a simple process f we define the integral in the usual way. Suppose $f(s) = \sum_\iota x\chi_I(s)$. Then for $E \in \mathscr{F}$,

$$\int_E f\, d\mu = \sum_\iota x_i \mu(E \cap I_\iota).$$

One can verify that the integral of a simple process is independent of its representation as a sum of elementary processes. We denote the set of simple processes on $E \in \mathscr{F}$ by $\mathscr{S}(E)$. We draw attention to the following, which is a counterpart to Bartle's Lemma 1.

2.4. LEMMA. (i) *For $E \in \mathscr{F}$, $\mathscr{S}(E)$ is a linear space under pointwise operations.*

(ii) *For a fixed $E \in \mathscr{F}$ the map $\mathscr{S}(\mathbf{T}) \ni f \to \int_E f\, d\mu \in z$ is a linear mapping.*

(iii) *For a fixed $f \in \mathscr{S}(\mathbf{T})$ the map $\mathscr{F} \ni E \to \int_E f\, d\mu \in z$ is an additive function.*

(iv) *If $f \in \mathscr{S}(\mathbf{T})$ and $(\exists M \in \mathbb{R}^+)$: $(\forall s \in E \in \mathscr{F}) |f(s)| < M$ then $|\int_E f\, d\mu| \leqslant M\|E\|^b$.*

Proof. (i) Note that processes form a linear space. The rest is a routine argument. Parts (ii) and (iii) may be verified by copying the proofs from classical measure theory, e.g., [10].

(iv) This follows directly from the definition of the integral and the definition of $\|\cdot\|^b$ (multiply by M/M). Q.E.D.

We are now able to mimic the construction of the integral given in Section 2 of [6]. Following Bartle we make the following definition.

2.5. Definition. (i) Let f, $g_1, g_2, \ldots$ be functions $\mathbf{T} \to X$. We say that (g_n) converges to f μ^b- (read this as μ-flat) almost everywhere on a set $E \in \mathscr{F}$ if there is a set $A \subseteq E$ with $\|A\|^b = 0$ and $\forall s \in E \backslash A$, $g_n(s) \to f(s)$ in norm.

(ii) Let $f, g_1, g_2, \ldots$ be functions $\mathbf{T} \to X$. We say that (g_n) converges to f μ^b-almost uniformly on $E \in \mathscr{F}$ if $\forall \varepsilon > 0$ $\exists A \subseteq \mathbf{T}$: $\|A\|^b < \varepsilon$ and the convergence is uniform on $E \backslash A$.

(iii) Let $f, g_1, g_2, \ldots$ be functions $\mathbf{T} \to X$. We say that (g_n) converges to f in μ^b-measure on a set $E \in \mathscr{F}$ if $\forall \varepsilon > 0$,

$$\|\{s \in E : |f(s) - g_n(s)| \geqslant \varepsilon\}\|^b \to 0 \qquad \text{as } n \to \infty.$$

(iv) A function $f: \mathbf{T} \to X$ is said to be μ^b-measurable if there is a sequence of simple processes converging to f in μ^b-measure on $\mathbf{T}$.

We shall use the abbreviations a.e. (almost everywhere) and a.u. (almost uniform). The various notions relate as one might expect. We have, for instance,

2.6. Proposition. *Let $E \in \mathscr{F}$ and $f, f_1, f_2, \ldots$ be μ^b-measurable.*

(i) *If $f_n \to f$ μ^b- a.u. on E then $f_n \to f$ in μ^b-measure on E.*

(ii) *If $f_n \to f$ μ^b-a. u. on E then $f_n \to f$ μ^b-a.e. on E.*

Proof. Use monotonicity and subadditivity of $\|\cdot\|^b$ and copy the usual proofs of these results. Q.E.D.

If we have countable subadditivity of $\|\cdot\|^b$ then we can recover another familiar result.

2.7. Proposition. *Let $f, f_1, f_2, \ldots$ be processes and let $\|\cdot\|^b$ be countably subadditive on the power set of $\mathbf{T}$. If (f_n) converges to f in μ^b-measure, then some subsequence converges to f μ^b-a.e.*

Proof. If $f_n \to f$ in μ^b-measure then $\forall m \in \mathbb{N}$ $\exists N_m : \forall n \geqslant N_m$ $\|\{s : |f(s) - f_n(s)| \geqslant 2^{-m}\}\|^b < 2^{-m}$. Consider (f_{N_m}). Let $H = \{s : f_{N_m}(s) \nrightarrow f(s)\}$. If $s \in H$ $\exists 1 \geqslant \varepsilon(s) > 0$: $\forall m$ $\exists k \geqslant m$: $|f_{N_k}(s) - f(s)| \geqslant \varepsilon(s)$. So for each m such that $\varepsilon(s) > 2^{-m}$ there is k such that $|f_{N_k}(s) - f(s)| \geqslant \varepsilon(s) > 2^{-m} \geqslant 2^{-k} \geqslant 2^{-N_k}$. Hence $s \in \{s : |f_{N_k}(s) - f(s)| \geqslant 2^{-N_k}\}$. So for each m such that $\varepsilon(s) > 2^{-m}$, $s \in \bigcup_{k \geqslant m} \{s : |f_{N_k}(s) - f(s)| \geqslant 2^{-N_k}\} = A_m$; i.e., $\forall s \in H$ $\exists M : \forall m \geqslant M$ $s \in A_m$. But (A_m) is a decreasing sequence of sets so that $\forall m$ $(H \subseteq A)$. Hence

$$\forall m,\ \|H\|^b \leqslant \|A_m\|^b \leqslant \sum_{k \geqslant m}^{\infty} \|\{s\colon |f_{N_k}(s) - f(s)| \geqslant 2^{-N_k}\}\|^b$$

$$< \sum_{k=m}^{\infty} 2^{-N_k} < 2^{-(m-1)}$$

using monotonicity and countable subadditivity. So H is a $\|\cdot\|^b$ null set. Q.E.D.

2.8. COROLLARY. *If $\|\cdot\|^b$ is countably subadditive and $f\colon \mathbf{T} \to X$ is μ^b-measurable then for μ^b- almost every $s \in \mathbf{T}$, $f(s) \in \bar{X}_s$, where $\bar{X}_s$ denotes the closure of X_s in X; that is, f is a process μ^b-a.e.*

Proof. By 2.5(iv) there is a sequence of simple processes converging to f in μ^b-measure. By 2.7 some subsequence converges μ^b-almost everywhere. The result follows because $\bar{X}_s$ is closed and the sequence consist of processes. Q.E.D.

2.9. *Control Measure.* It is natural to ask under what conditions countable subadditivity of $\|\ \|_\mu^b$ is assured. Our work in [2–4] has indicated that countable subadditivity and some other very desirable properties of $\|\cdot\|_\mu^b$ will hold if there is a "control measure" for μ. To be precise, that there is an extended real valued countably additive set function, ν, defined on $\mathscr{F}$ with the property that $\nu(E) \to 0 \Rightarrow \|E\|_\mu^b \to 0$ for $E \in \mathscr{F}$. The existence of such a measure allows us to extend $\|\cdot\|_\mu^b$ from $\mathscr{F}$ to the ν-measurable subsets of $\mathbf{T}$ in a manner different from that of 2.2. The result of this is to widen the class of integrable functions and we shall indicate why a little later. First we look at countable subadditivity.

2.10. PROPOSITION. *Suppose that μ is finitely additive on $\mathscr{F}$. Let ν be a countably additive positive real valued set function on $\mathscr{F}$ and $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ some function continuous at $0 \in \mathbb{R}$. Suppose further that $\forall E \in \mathscr{F}$, $\|E\|^b \leqslant f(\nu(E))$. Then $\|\cdot\|^b$ is countably subadditive on $\mathscr{F}$.*

Proof. Let $E, E_1, E_2, \ldots$ be elements of $\mathscr{F}$ with $E = \bigcup_n E_n \in \mathscr{F}$. As $E = (E \backslash \bigcup_{n=1}^m E_n) \cup (\bigcup_{n=1}^m E_n)$ then by finite additivity

$$\|E\|^b \leqslant \left\| E_n \Big\backslash \bigcup_{n=1}^m E_n \right\|^b + \left\| \bigcup_{n=1}^m E_n \right\|^b,$$

which by hypothesis is

$$\leqslant f\left(\nu\left(E_n \Big\backslash \bigcup_{n=1}^m E_n\right)\right) + \sum_{n=1}^m \|E_n\|^b$$

$$\leqslant f\left(\nu\left(E_n \Big\backslash \bigcup_{n=1}^m E_n\right)\right) + \sum_{n=1}^{\infty} \|E_n\|^b.$$

Now $E_n \backslash \bigcup_{n=1}^{m} E_n \downarrow \varnothing$ as $m\uparrow$, $v(\cdot)$ is continuous from above at $\varnothing$ and f is continuous at zero. So $\|\cdot\|^b$ is countably subadditive on $\mathscr{F}$. Q.E.D.

Remarks on Proposition 2.10. (i) Countable subadditivity can quickly break down outside of $\mathscr{F}$. If $X = Y = Z = X_\alpha = [0, T]$ and μ is Lebesgue measure on $\mathscr{F}$ then $E = \mathbb{Q} \cap [0, T]$ is countable, $E = \{x_n : n \in \mathbb{N}\}$, say. Clearly $E = \bigcup_n \{x_n\}$ and $\|\{x_n\}\|_\mu^b = 0$ thus $\sum_n \|\{x_n\}\|_\mu^b = 0$. However, the smallest element of $\mathscr{F}$ containing E is $[0, T]$; thus $\|E\|_\mu^b = T$. We can see that $\| \ \|_\mu^b$ is not countably subadditive outside of $\mathscr{F}$.

(ii) We shall be concerned with the case when v is continuous with respect to Lebesgue measure and $f(x) = Cx^{1/2}$, C some positive constant. Clearly v is a control measure for μ.

Now we consider extending $\|\cdot\|^b$ from $\mathscr{F}$ to all subsets of $\mathbf{T}$ when a control measure exists. We will consider the particular case when the control measure is λ, Lebesgue measure. In our examples in Section 4 the control measure is often different from λ but it is always λ-continuous. The discussion should make it clear that the use of λ is not essential.

2.11. Definition. (i) Let $O(\mathscr{F}) = \{E : E = \bigcup_n E_n,\ E_n\uparrow,\ E_n \in \mathscr{F}\}$, i.e., outer sets in $[0, T]$.

(ii) For $E \in O(\mathscr{F})$ define $\|E\|_\mu^{b_\sigma} = \lim_n \|E_n\|_\mu^b$, where $(E_n) \subset \mathscr{F}$, and $E_n \uparrow E$.

2.12. Proposition. *For $E \in O(\mathscr{F})$, $\|E\|_\mu^{b_\sigma}$ is well defined. If $E \in \mathscr{F}$ then $\|E\|_\mu^b = \|E\|_\mu^{b_\sigma}$. Moreover $\| \ \|_\mu^{b_\sigma}$ is monotone and countably subadditive on $O(\mathscr{F})$.*

Proof. If $E_n \uparrow E \uparrow F_n$ and $E_n, F_n \in \mathscr{F}$, $n = 1, 2, \ldots$, then, as $F_n = (F_n \cap E_n) \cup (F_n \backslash E_n)$ and $\|\cdot\|_\mu^b$ is monotone and subadditive, we have

$$\|F_n \cap E_n\|_\mu^b \leqslant \|F_n\|_\mu^b \leqslant \|F_n \cap E_n\|_\mu^b + \|F_n \backslash E_n\|_\mu^b.$$

But $E \backslash E_n \supseteq F_n \backslash E_n$ and $E_n \uparrow E$, so $F_n \backslash E_n \downarrow \varnothing$ and hence $\lambda(F_n \backslash E_n) \downarrow 0$. Thus $\|F_n \backslash E_n\|_\mu^b \to 0$. So $\lim_n \|F_n \cap E_n\|_\mu^b = \lim_n \|F_n\|_\mu^b$. Hence $\lim_n \|F_n\|_\mu^b = \lim_n \|E_n\|_\mu^b$.

It is now clear that if $E \in \mathscr{F}$ then $\|E\|_\mu^b = \|E\|_\mu^{b_\sigma}$ by taking the sequence $E_n = E$. If $E, F \in O(\mathscr{F})$ and $E \subseteq F$ and $E_n \uparrow E$, $F_n \uparrow F$ then $E_n \cup F_n \in \mathscr{F}$ and $E_n \cup F_n \uparrow F$. So as $E_n \subseteq E_n \cup F_n$ then $\|E\|_\mu^{b_\sigma} = \lim_n \|E_n\|_\mu^b \leqslant \lim_n \|E_n \cup F_n\|_\mu^b = \|F\|_\mu^{b_\sigma}$ by the first part. It is clear that $\| \ \|_\mu^{b_\sigma}$ is finitely subadditive. Suppose $E_n \in O(\mathscr{F})$, E_n disjoint, and $E = \bigcup_n E_n$. Then $E \in O(\mathscr{F})$. Suppose

$E_n = \bigcup_m G_{mn}$, where $G_{mn} \in \mathscr{F}$ and $G_{mn} \uparrow E_n$ as $m \uparrow$. Let $H_l = \bigcup_{n=1}^{l} G_{ln}$. Then $H_l \uparrow E$ as $l \uparrow$ and, by disjointness of the E_n's,

$$\|H_l\|_\mu^b \leqslant \sum_{n=1}^{l} \|G_{ln}\|_\mu^b \leqslant \sum_{n=1}^{l} \|E_n\|_\mu^{b_\sigma} \leqslant \sum_{n=1}^{\infty} \|E_n\|_\mu^{b_\sigma}.$$

Taking the limit on the left-hand side gives the result. Q.E.D.

2.13. Definition. Let $A \subseteq \mathbf{T} \cdot \|A\|_\mu^{b_\sigma} = \inf\{\|E\|_\mu^{b_\sigma} : E \supseteq A, \quad E \in O(\mathscr{F})\}$. With this extension of $\|\cdot\|_\mu^b$ we can define μ^{b_σ}-null sets, convergence μ^{b_σ}-*a.u.*, and convergence in μ^{b_σ}-measure just as in 2.5 but with $\|\cdot\|_\mu^{b_\sigma}$ replacing $\|\cdot\|_\mu^b$. The results corresponding to 2.6, 2.7, and 2.8 will hold.

For our examples in Section 4 we will need the following results and their variants with λ replaced by some λ continuous measure.

2.14. Proposition. *Suppose that* $E \in \mathscr{F} \Rightarrow \|E\|_\mu^b \leqslant \lambda(E)^{1/2}$. *Then for any Lebesgue-measurable set* $F \subseteq \mathbf{T}$, $\|F\|_\mu^{b_\sigma} \leqslant \lambda(F)^{1/2}$.

Proof. Let $E \in O(\mathscr{F})$ and $\mathscr{F} \in E_n \uparrow E$; then

$$\|E\|_\mu^{b_\sigma} = \lim_n \|E_n\|_\mu^b \leqslant \lim_n \lambda(E_n)^{1/2} = \lambda(E)^{1/2}.$$

So

$$\|F\|_\mu^{b_\sigma} = \inf\{\|E\|_\mu^{b_\sigma} : E \in O(\mathscr{F}),\ E \supseteq F\} \leqslant \inf_{\substack{E \supseteq F \\ E \in O(\mathscr{F})}} \lambda(E)^{1/2} = \lambda(F)^{1/2}. \quad \text{Q.E.D.}$$

2.15. Proposition. $\|\cdot\|_\mu^{b_\sigma}$ *is countalby subadditive on the power set of* $\mathbf{T}$.

Proof. Let $\varepsilon > 0$. Let $E, E_n, n = 1, 2, \ldots$, be subsets of $\mathbf{T}$ with $\{E_n\}$ disjoint. Let $F_n \in O(\mathscr{F})$, $F_n \supseteq E_n$, $\|F_n\|_\mu^{b_\sigma} < \|E_n\|_\mu^{b_\sigma} + \varepsilon/2^n$. Let $F = \bigcup_n F_n$. Note that $\{F_n\}$ is not necessarily a disjoint family. We have $F \in O(\mathscr{F})$, $F \supseteq E$. Now if $G_1 = F_1$, $G_2 = F_2 \backslash F_1$, $G_3 = F_3 \backslash F_2 \backslash F_1$, and so on, then $\{G_n\}$ is a disjoint family of outer sets with union F. By 2.12 then

$$\|E\|_\mu^{b_\sigma} \leqslant \|F\|_\mu^{b_\sigma} \leqslant \sum_{n=1}^{\infty} \|G_n\|_\mu^{b_\sigma} \leqslant \sum_{n=1}^{\infty} \|F_n\|_\mu^{b_\sigma}$$

$$\leqslant \sum_{n=1}^{\infty} (\|E_n\|_\mu^{b_\sigma} + \varepsilon/2^n). \quad \text{Q.E.D.}$$

If μ is "controlled" by λ (e.g., as in 2.14) it follows that any λ-measurable function is also μ^{b_σ}-measurable because convergence in λ-measure implies convergence in μ^{b_σ}-measure. If we did not extend $\|\cdot\|_\mu^b$ to $\|\cdot\|_\mu^{b_\sigma}$ when a control measure was present then our integration theory would be unnecessarily limited, as the following example illustrates. Suppose that, in the notation of Section 1, $[0, T] = X = Y = Z = X_\alpha$ for $\alpha \in [0, T]$ and that μ is Lebesgue measure on $\mathscr{F}$. Let $g = \chi_{[0,T] \cap \mathbb{Q}}$, where $\mathbb{Q}$ denotes the rational numbers. Since $\mu = \lambda$ on $\mathscr{F}$ it follows that $\|A\|_\mu^{b_\sigma} = \lambda(A)$ for Lebesgue-measurable $A \subseteq [0, T]$ and hence that g is μ^{b_σ}-measurable. However g is *not* μ^b-measurable, as can be seen by examining a sequencce (g_n) of simple processes (step functions!) converging to g in μ^b-measure. For if $g_n \to g$ in μ^b-measure then there is $N \in \mathbb{N}$ such that $n \geqslant N$ gives $\|\{|g_n - g| > \frac{1}{8}\}\|_\mu^b < \frac{1}{8}$. So there is $F \in \mathscr{F}$ with $F \supseteq \{|g_N - g| > \frac{1}{8}\}$ and $\lambda(F) < \frac{1}{8}$. We consider $|g_N - g|$ on $[0, T] \backslash F$. We have $|g_N - g| \leqslant \frac{1}{8}$ on $[0, T] \backslash F$. Now g_N is just a step function and $[0, T] \backslash F$ is a finite union of intervals at least one of which has positive λ-measure. It follows that there is an interval $I \subseteq [0, T] \backslash F$ with $\lambda(I) > 0$ on which g_N is constant and $|g_N - g| \leqslant \frac{1}{8}$, which is impossible because $I \cap \mathbb{Q} \neq \varnothing$ and $I \backslash \mathbb{Q} \neq \varnothing$.

In the next section we discuss integration theory. We will write $\|E\|^b$ for the belated semivariation of a set $E \subseteq \mathbf{T}$, which may be defined either as in 2.2 or, if a control measure exists, as in 2.13.

3. The Belated Integral: General Theory

In this section we prove the counterparts to Bartle's theorems 1–5. The proofs are taken from Bartle where possible.

3.1. Definition. (i) Let $f: \mathbf{T} \to X$ be μ^b-measurable and let (f_n) be a sequence of simple processes converging to f in μ^b-measure. We say that f is μ^b-integrable over $\mathbf{T}$ if (f_n) can be chosen so that

(a) $\forall \varepsilon > 0\ \exists \delta > 0$: $E \in \mathscr{F}$ and $\|E\|^b < \delta$ then $\forall n\, |\int_E f_n\, d\mu| < \varepsilon$.

(b) $\forall \varepsilon > 0\ \exists F \in \mathscr{F}$ with $\|F\|^b < \infty$ and if $G \in \mathscr{F}$, $G \subseteq \mathbf{T} \backslash F$, then $\forall n\, |\int_G f_n\, d\mu| < \varepsilon$.

(ii) Let $E \in \mathscr{F}$ and f be μ^b-measurable. We say that f is μ^b-integrable over E if $f \cdot \chi_E$ is μ^b-integrable over $\mathbf{T}$.

3.2. *Remark.* We shall say that a sequence of simple processes (f_n) *defines* a function f if $f_n \to f$ in μ^b-measure. If in addition f happens to be integrable we will suppose, unless otherwise stated, that (f_n) satisfies the conditions a, b of Definition 3.1(i). Note that if $\|\cdot\|^b$ is defined as in 2.13 then the class of measurable processes is larger than that given by 2.2. The class of integrable functions given by 3.1 is then, accordingly, larger. Integration over sets outside of $\mathscr{F}$ will be dealt with in another publication.

3.3. THEOREM. *Let f be μ^b-integrable and let (f_n) be a sequence of simple processes defining f. For each $E \in \mathscr{F}$, $(\int_E f_n\, d\mu)$ is a Cauchy sequence in Z. We denote the limit by $\int_E f\, d\mu^b$ and call it the μ-belated integral of f over E. The limit exists uniformly for $E \in \mathscr{F}$. The μ-belated integral of f over R is independent of the sequence of simple processes that define f.*

Proof. Since (f_n) converges to f in μ^b-measure it is Cauchy in μ^b-measure, i.e., $\forall r > 0$, $\forall t > 0$, $\exists N(t) \in \mathbb{N}$: $m, n \geqslant N(t)$, then $\|\{s\colon |f_n(s) - f_m(s)| \geqslant r\}\|^b < t$. Let $\varepsilon > 0$ and choose $\delta > 0$ as in 3.1(i, a) and F_ε as in 3.1(i. b). Taking $r = \varepsilon/(1 + \|F_\varepsilon\|^b)$ and $t = \delta$ our first remark indicates that there is $N \in \mathbb{N}$ such that if $m, n \geqslant N$ are fixed then the set $G = \{s\colon |f_m(s) - f_n(s)| \geqslant \varepsilon/(1 + \|F_\varepsilon\|^b)\} \in \mathscr{F}$ and $\|G\|^b < \delta$. Moreover if $s \notin G$ then $|f_n(s) - f_m(s)| < \varepsilon/(1 + \|F_\varepsilon\|^b)$. Let $E \in \mathscr{F}$. We can write E as a disjoint union of sets in $\mathscr{F}$ as follows: $E = (E \cap G) \cup ((E \backslash G) \cap F_\varepsilon) \cup (E \backslash (G \cup F_\varepsilon))$. Hence by 2.4(iii) and the triangle inequality,

$$\left|\int_E f_n\, d\mu - \int_E f_m\, d\mu \leqslant \left|\int_{E \cap G} f_n\, d\mu\right| + \left|\int_{E \cap G} f_m\, d\mu\right| + \left|\int_{E \backslash (G \cap F_\varepsilon)} f_n\, d\mu\right|\right.$$

$$+ \left|\int_{E \backslash (G \cup F_\varepsilon)} f_m\, d\mu\right| + \left|\int_{(E \backslash G) \cap F_\varepsilon} (f_n - f_m)\, d\mu\right|$$

Now $\|E \cap G\|^b \leqslant \|G\|^b < \delta$ and so the terms involving $E \cap G$ are strictly less than ε (3.1(i), a). Also $E \backslash (G \cup F_\varepsilon) \subseteq E \backslash F_\varepsilon$; hence the terms involving $E \backslash (G \cup F_\varepsilon)$ are strictly less than ε (3.1(i), b). Finally by 2.4(iv) and the fact that $(E \backslash G) \cap F_\varepsilon \subseteq T \backslash G$

$$\left|\int_{(E \backslash G) \cap F_\varepsilon} (f_n - f_m)\, d\mu\right| \leqslant \left(\frac{\varepsilon}{1} + \|F_\varepsilon\|^b\right) \cdot \|(E \backslash G) \cap F_\varepsilon\|^b < \varepsilon$$

by monotonicity of $\|\cdot\|^b$. This shows that $(\int_E f_n d\mu)$ is Cauchy in Z and that the limit exists uniformly for $E \in \mathscr{F}$. Suppose now that (f_n) and (g_n) are sequences of simple processes defining f. It is easily verified that the sequence $(f_1, g_1, f_2, g_2, \ldots)$ satisfies 3.1(i), a, b and hence that $\int_E f\, d\mu^b = \lim_n \int_E f_n\, d\mu = \lim_n \int_E g_n\, d\mu$. Q.E.D.

Remark. In 2.5 we defined convergence in μ^b-measure for *functions* $g\colon \mathbf{T} \to X$. Now in 3.1 we take a μ^b-measurable f defined by simple *processes* (f_n). Definition 3.1 still makes sense if we replace f by some function g which is the limit in μ^b-measure of simple *functions* (g_n) satisfying 3.1(a, b). One might think that we could extend the μ^b-integral to such g using 3.3 appropriately modified. This is not the case, for 3.3 fails if the (g_n) are not processes. For an example of this, see Section 4.2.

3.4. THEOREM. (i) *If $E \in \mathscr{F}$ then the set of functions which are μ^b-integrable over E form a linear space under pointwise operations. We denote this space by $I^b(E)$. The map*

$$I^b(E) \ni f \to \int_E f\,d\mu^b \in Z$$

is a linear mapping.

(ii) *If $f \in I^b(\mathbf{T})$ the map*

$$\mathscr{F} \in E \to \int_E f\,d\mu^b \in Z$$

is an additive function.

(iii) *If $f \in I^b(\mathbf{T})$ then*

$$\lim_{\|E\|^b \to 0} \left|\int_E f\,d\mu^b\right| = 0.$$

(iv) *If $f \in I^b(\mathbf{T})$, $\forall \varepsilon > 0$, $\exists E_\varepsilon \in \mathscr{F}: G \subseteq \mathbf{T} \backslash E_\varepsilon$ then*

$$\left|\int_G f\,d\mu^b\right| < \varepsilon.$$

Proof. Parts (i) and (ii) use 2.4 and some routine methods. To prove (iii) we use the uniformity of the limit established in 3.3. Let $\varepsilon > 0$. If (f_n) define f then for a large enough fixed $n \in \mathbb{N}$ we have

$$\forall E \in \mathscr{F}, \left|\int_E f\,d\mu^b\right| \leqslant \left|\int_E f\,d\mu^b - \int_E f_n\,d\mu\right| + \left|\int_E f_n\,d\mu\right|$$
$$< \frac{\varepsilon}{2} + \left|\int_E f_n\,d\mu\right|.$$

Let M be a bound for $\{|f_n(s)|: s \in \mathbf{T}\}$. By 2.4(iv), $\forall E \in \mathscr{F}$, $|\int_E f\,d\mu^b| < \varepsilon/2 + M\|E\|^b$. Adjusting $\|E\|^b$ appropriately gives $|\int_E f\,d\mu^b| < \varepsilon$ whenever $\|E\|^b$ is sufficiently small. On the other hand if $\delta > 0$ is a bound for $\{|\int_E f_n\,d\mu|: n \in \mathbb{N}\}$ then we have $|\int_E f\,d\mu^b| < \varepsilon/2 + \delta$, which proves (iv) when 3.1 is taken into account. Q.E.D.

3.5. DEFINITION. Let $f: \mathbf{T} \to X$. We say that f is μ^b essentially bounded on $E \subseteq \mathbf{T}$ if

$$\operatorname*{ess\,sup}_{s \in E} |f(s)| \stackrel{\text{def}}{=} \inf_N (\sup_{s \in E \backslash N} |f(s)|) < \infty,$$

where the infimum is taken over all μ^b-null sets.

3.6. THEOREM. *A μ^b essentially bounded μ^b-measurable function f is integrable over each $E \in \mathscr{F}$ with $\|E\|^b < \infty$. Moreover*

$$\left|\int_E f\,d\mu^b\right| \leqslant \operatorname*{ess\,sup}_{s \in E} |f(s)| \cdot \|E\|^b.$$

Proof. Let (f_n) be a sequence of simple processes that define f. Let $k \in \mathbb{N}$ be fixed and $M = \operatorname{ess\,sup}_{s \in E} |f(s)|$. Consider the functions h_n, $n < 1, 2, \ldots$, where

$$\begin{aligned} h_n(s) &= f_n(s) && \text{if } \ |f_n(s)| \leqslant M + 1/k \\ &= \frac{f_n(s)}{f_n(s)|} \cdot (M + 1/k) && \text{if } \ |f_n(s)| > M + 1/k. \end{aligned}$$

Note that (h_n) is a sequence of *simple processes*. There is a μ^b-null set B_k such that $\{s \in E\colon |f(s)| \geqslant M + 1/k\} \subseteq B_k$. So $\{s \in E\colon |f_n(s)| > M + 1/k\} \subset \{s \in E\colon \ |f(s) - f_n(s)| \geqslant 1/2k\} \cup B_k$. Hence $\{s \in E\colon |h_n(s) - f(s)| \geqslant \varepsilon\} \subseteq \{s \in E\colon |f_n(s) - f(s)| \geqslant \varepsilon\} \cup B_k \cup \{s \in E\colon \ |f(s) - f_n(s)| \geqslant 1/2k\}$. So $h_n \to f$ in μ^b-measure on E. So we can assume that our sequences of simple processes is uniformly bounded. This shows that 3.1(i, a) is satisfied and as $\|E\|^b < \infty$, 3.1(i, b) is automatically satisfied (take E!). Finally, taking n large enough and using 3.3 and 2.4(iv),

$$\left|\int_E f\,d\mu^b\right| \leqslant \left|\int_E (f - h_n)\,d\mu^b\right| + \left|\int_E h_n\,d\mu^b\right| \leqslant \varepsilon + (M + 1/k) \cdot \|E\|^b$$

for a given $\varepsilon > 0$. As this holds for each $k \in \mathbb{N}$ the result follows. Q.E.D.

The next result concludes our section on general theory.

3.7. THEOREM. *Let $f\colon \mathbf{T} \to X$ and $f_1, f_2, \ldots$ be elements of $I^b(\mathbf{T})$. Suppose that*

(i) *(f_n) converges to f in μ^b-measure.*

(ii) $\forall \varepsilon > 0\ \exists \delta > 0$: *$E \in \mathscr{F}$ and $\|E\|^b < \delta$ then* $\forall n\, |\int_E f_n\,d\mu^b| < \varepsilon$

(iii) $\forall \varepsilon > 0\ \ \exists F \in \mathscr{F}$: $\|F\|^b < \infty$ *and if* $G \in \mathscr{F}$ *and* $G \subseteq \mathbf{T} \backslash F$ *then* $\forall n\, |\int_G f_n\,d\mu^b| < \varepsilon$.

Then $f \in I^b(\mathbf{T})$ and for $E \in \mathscr{F}$

$$\int_E f\,d\mu^b = \lim_n \int_E f_n\,d\mu^b.$$

Moreover this limit is uniform for $E \in \mathscr{F}$.

Proof. For each $n \in \mathbb{N}$ there is a simple process g_n such that

$$\|\{s \in \mathbf{T}\colon |f_n(s) - g_n(s)| \geqslant 2^{-n}\}\|^b < 2^{-n} \text{ and } \left|\int_E f_n \, d\mu^b - \int_E g_n \, d\mu\right| < 2^{-n}$$

uniformly for $E \in \mathscr{F}$ (3.3). Now,

$$\begin{aligned}\{s \in \mathbf{T}\colon |g_n(s) - f(s)| \geqslant 2\varepsilon\} \subseteq \{s \in \mathbf{T}\colon |g_n(s) - f_n(s)| \geqslant \varepsilon\} \\ \cup \{s \in \mathbf{T}\colon |f_n(s) - f(s)| \geqslant \varepsilon\}\end{aligned}$$

and so $g_n \to f$ in μ^b-measure. So f is μ^b-measurable. Now the sequence (g_n) satisfies 3.1(a) because for $E \in \mathscr{F}$

$$\forall n, \left|\int_E g_n \, d\mu\right| < \left|\int_E f_n \, d\mu^b + 2^{-n}\right| \tag{*}$$

and (f_n) satisfy (ii) of the hypothesis. To see that 3.1(b) holds we choose $F \in \mathscr{F}$ in 3.7(ii) for $\varepsilon/2$ and using (*) discard finitely many terms of the sequence (g_n) so that the remaining terms satisfy 3.1(b). This shows that f is μ^b-integrable and 3.3 indicates that $\forall E \in \mathscr{F}$

$$\int_E f \, d\mu^b = \lim_n \int_E g_n \, d\mu = \lim_n \int_E f_n \, d\mu. \qquad \text{Q.E.D.}$$

3.8. *Remark.* Each of the results of this section has its left-belated counterpart.

If we allow the degenerate case $\forall \alpha \; X_\alpha = X$ then the belated semivariation is just the semivariation (with respect to X). And the belated integral is just the Bartle integral associated with $(\mathbf{T}, \mathscr{F}, \mu)$.

4. Examples of Belated Integrals

4.1. *Contraction Integral in a Probability Gauge Space* [4]

Let $(\mathscr{A}_\alpha)$, $\alpha \in \mathbf{T}$, be an increasing family of finite von Neumann algebras acting on a fixed Hilbert space $\mathscr{H}$ and which satisfy:

(i) if $\alpha_1 \leqslant \alpha_2$ then $\mathscr{A}_{\alpha_1}$ is a von Neumann subalgebra of $\mathscr{A}_{\alpha_2}$;

(ii) $\mathscr{A}_{\mathbf{T}} = (\bigcup_\alpha \mathscr{A}_\alpha)''$ is finite;

(iii) $\bigcap_{\beta > \alpha} \mathscr{A}_\beta = \mathscr{A}_\alpha$.

Let ϕ be a faithful normal finite trace on $\mathscr{A}_{\mathbf{T}}$ with $\phi(I) = 1$. We write $L^p(\mathscr{A}_\alpha)$, $1 \leqslant p < \infty$, $\alpha \in \mathbf{T}$, to denote the non-commutative Lebesgue spaces [13] associated with the pair $(\mathscr{A}_\alpha, \phi)$. We note that $L^p(\mathscr{A}_{\alpha_1}) \subseteq L^p(\mathscr{A}_{\alpha_2}) \subseteq L^p(\mathscr{A}_{\mathbf{T}})$ for $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \mathbf{T}$ and that the conditional expectation $M_\alpha: L^p(\mathscr{A}_{\mathbf{T}}) \to L^p(\mathscr{A}_\alpha)$ exists and satisfies the properties listed in [12]. A family (X_α), $\alpha \in \mathbf{T}$ of elements of $L^1(\mathscr{A}_{\mathbf{T}})$ is a *process* if $X_\alpha \in L^1(\mathscr{A}_\alpha)$. A process (X_α) is a martingale if $M_{\alpha_1}(X_{\alpha_2}) = X_{\alpha_1}$, $\alpha_1 \leqslant \alpha_2$.

We note that the elements of $L^p(\mathscr{A}_{\mathbf{T}})$ are a subspace of the set $M(\mathscr{A}_{\mathbf{T}})$ of all densely defined closed operators affiliated with $\mathscr{A}_{\mathbf{T}}$. Under *strong* sum and *strong* product $M(\mathscr{A}_{\mathbf{T}})$ forms a topological *-algebra. Hereafter we use sum and product in this sense without comment. The spaces $L^p(\mathscr{A}_\alpha)$ are generalisations of the L^p spaces of measure theory and they share many of their important features (see [13]). Let us now relate this to the context described in Section 1. We take $(\mathscr{A}_\alpha)$ to be our "nest" of normed spaces (X_α) in Section 1. For Y and Z we take $L^2(\mathscr{A}_{\mathbf{T}})$. We note immediately that the product of $A \in \mathscr{A}_{\mathbf{T}}$ and $X \in L^2(\mathscr{A}_{\mathbf{T}})$ satisfies $\|AX\|_2 \leqslant \|A\|_\infty \|X\|_2$ and is bilinear. Let (x_α) be an L^2-martingale *bounded in* L^2. Since (x_α) is bounded it converges in L^2 to some $x_{\mathbf{T}} \in L^2(\mathscr{A}_{\mathbf{T}})$ as $\alpha \to T$ [1]. For $0 \leqslant s \leqslant t \leqslant T$ define $dx((s,t]) = x_t - x_s$. This function extends to sets that are finite (disjoint) unions of left open right closed intervals. Let θ denote the field of subsets of $(0, \mathbf{T}]$ generated by such sets.

4.1. Proposition. *The set function dx has finite right-belated semi-variation with respect to* $(\mathscr{A}_\alpha)$ *on* $(0, T]$.

Proof. First let $0 \leqslant s \leqslant t$ and $A \in \mathscr{A}_s$. Then

$$\begin{aligned}\|A(x_t - x_s)\|_2^2 &= \phi((x_t - x_s)^* A^* A(x_t - x_s)) \\ &\leqslant \|A\|_\infty^2 \, \phi((x_t - x_s)^*(x_t - x_s))\end{aligned}$$

by Hölder's inequality. Using the martingale property we get $\|A(x_t - x_s\|_2^2 \leqslant \|A\|_\infty^2 \phi(|x_t|^2 - |x_s|^2)$. If we now consider a sum such as $\sum_j A_j(x_{t_j} - x_{t_{j-1}})$ with $A_j \in \mathscr{A}_{t_{j-1}}$ then $\|\sum_j A_j(x_{t_j} - x_{t_{j-1}})\|_2^2 \leqslant \sum_j \|A_j\|_\infty^2 \phi(|x_{t_j}|^2 - |x_{t_{j-1}}|^2)$ because the "cross terms" in $\phi\{(\sum_j A_j(x_{t_j} - x_{t_{j-1}})^*(\sum_i A_i(x_{t_i} - x_{t_{i-1}}))\}$ vanish when the conditional expectation $M_{\max\{t_{i-1}, t_{j-1}\}}$ is taken inside the trace ϕ. The last inequality may be rewritten

$$\left\| \int f \, dx \right\|_2^2 \leqslant \int \|f\|_\infty^2 \, d|\langle X \rangle|, \tag{**}$$

where $f(s) = \sum_j A_j \chi^{(s)}_{(t_{j-1}, t_j]}$ and $d|\langle X \rangle|$ is the Stieltjes measure obtained from $s \to \phi(|x_s|^2)$. Taking the supremum in the manner described in (1.02) shows that $\|(0, \mathbf{T}]\|_{dx}^b \leqslant \sqrt{\phi(|x|^2 - |x_0|^2)} = \|x_{\mathbf{T}} - x_0\|_2$. Q.E.D.

So $d|\langle X \rangle|$ is a control measure for $\|\cdot\|_{dx}^b$. The relation (**) looks identical to the contraction property of the non-commutative stochastic integral described in [4]. There is however a slight difference. In [4] the nest of von Neumann algebras is continuous, i.e., 4.1(iii) is augmented with $(\bigcup_{\beta<\alpha}\mathscr{A}_\beta)''=a_\alpha$. The effect of this is to make (x_α) continuous as a map $\mathbf{T}\to L^2(\mathscr{A}_\mathbf{T})$ and therefore $d\,|\langle X\rangle|$ is non-atomic. In the context of 4.1 we are only assured of right L^2 continuity of (x_α) so that $d\,|\langle X\rangle|$ may have atoms (because $\phi(|x_s|^2)$ may have jumps).

Bearing in mind the definition of dx^b-integrability, the fact that $d\,|\langle X\rangle|(\mathbf{T})<\infty$, and 2·9–2·15 we can see that the dx^b-integrable functions contain the processes in the set $\mathscr{P}$ (Definition 3.4 of (4)] and that the dx-belated integral and the non-commutative stochastic integral coincide.

4.2. *The Itô–Clifford Integral*

This is a particular case of that given in 4.1. However, the extra features of this situation allow far stronger results to be obtained. Our nest of von Neumann algebras is now the *Clifford filtration* $(\mathscr{C}_\alpha)$, $\alpha\in\mathbf{T}$ constructed as in [2]. The Clifford filtration is continuous in the sense described in 4.1. Once again we have the non-commutative L^p-spaces associated to $(\mathscr{C}_\alpha, m)$, where m is the trace obtained from the Fock vacuum (see [2]). We take the nest (X_α) of Section 1 to be $(L^2(\mathscr{C}_\alpha))$, $Y=\mathscr{C}_\mathbf{T}=(\bigcup_\alpha\mathscr{C}_\alpha)''$ and $X=Z=L^2(\mathscr{C}_\mathbf{T})$. Just as in Section 4.1 we will define a set function via a martingale. The field will be $\mathscr{F}$ as in Section 1. We refer the reader to Sections 1, 2, and 3 of [2] for a more complete explanation of the following. The Clifford algebra $\mathscr{C}_\mathbf{T}$ is generated by the fermion fields $\Psi(z)$ where $z\in L^2_\mathbb{C}(\mathbf{T}, ds)$. (For each $z\in L^2_\mathbb{C}(\mathbf{T}, ds)$ there is a bounded operator, $\Psi(z)$, acting on the antisymmetric Fock space associated with $L^2_\mathbb{C}(\mathbf{T}, ds)$.) If z is real valued $\Psi(z)$ is self-adjoint and the canonical anticommutation relations of quantum field theory yield the relation

$$\Psi(z)^2=\left(\int_\mathbf{T}|z|^2\right)I,\qquad \text{where } I \text{ is identity operator.}\tag{4.21}$$

Another very nice fact is that $(\Psi(z\chi_{[0,\alpha]}))$, $\alpha\in\mathbf{T}$ is a martingale. We consider the set function on $\mathscr{F}$ given by

$$d\Psi([s,t])=\Psi_t(z)-\Psi_s(z),\qquad 0\leqslant s\leqslant t\leqslant\mathbf{T}$$
$$d\Psi(\{s\})=0,$$

where $\Psi_t(z)=\Psi(z_{\chi[0,t]})$, $0\leqslant t\leqslant\mathbf{T}$.

4.22. Proposition. *The measure $d\Psi$ has finite belated semivariation with respect to $(L^2(\mathscr{C}_\alpha))$ over* $\mathbf{T}$.

Proof. As in 4.1 we consider a sum like $\sum_j A_j(\Psi_{t_j}-\Psi_{t_{j-1}})$ with $0\leqslant t_0\leqslant t_1\leqslant\cdots\leqslant t_n=\mathbf{T}$ and $A_j\in L^2(\mathscr{C}_{t_{j-1}})$. We have

$$\left\|\sum_j A_j(\Psi_{t_j}-\Psi_{t_{j-1}})\right\|_2^2=\sum_j m((\Psi_{t_j}-\Psi_{t_{j-1}})\,|A_j|^2\,(\Psi_{t_j}-\Psi_{t_{j-1}}))$$

(because "off diagonal" terms vanish as before)

$$=\sum_j m(|A_j|^2(\Psi_{t_j}-\Psi_{t_{j-1}})^2) \quad \text{because } m(\cdot) \text{ is a trace,}$$

$$=\sum_j m(|A_j|^2)\int_{t_{j-1}}^{t_j} |z(s)|^2\,ds \qquad \text{by (4.21)}$$

$$=\int_{\mathbf{T}} \|f(s)\|_2^2\,dv(s),$$

where we have written $f(s)=\sum_j A_j\chi_{[t_{j-1},t_j)}(s)$ and $v(E)=\int_E |z(s)|^2$ for $E\in\mathscr{F}$. Q.E.D.

We may rewrite the essential part of this result as

$$\left\|\int_{\mathbf{T}} f(s)\,d\Psi_s\right\|_2^2=\int_{\mathbf{T}} \|f(s)\|_2^2\,dv(s). \tag{4.23}$$

This is the isometry property of the Itô–Clifford integral. Taking the supremum as in (1.02) shows that $\|\cdot\|^b_{d\Psi}=\sqrt{v(\cdot)}$. As in 4.4 one can verify that the Itô–Clifford and $d\Psi^b$ integrals are the same and that 3.1 gives the same class of integrands as the Ito–Clifford theory.

The Clifford filtration provides us with an example of a measure with infinite semivariation (1.01) but finite belated semivariation. Let z in our discussion above be the function $s\to 1\in\mathbb{C}$, so that $\Psi_t=\Psi(\chi_{[0,t]})$. We have $\|[s,t]\|^b_{d\Psi}=(t-s)^{1/2}$. But if we let $t_j=Tj/n$, $1\leqslant j\leqslant n$, then put $x_j=(n/T)^{1/2}\Psi(\chi_{[t_{j-1},t_j]})$ and note that $\|x\|_\infty\leqslant 1$, we have

$$\left\|\sum_j x_j(\Psi_{t_j}-\Psi_{t_{j-1}})\right\|_2=\left\|\sum_j\left(\frac{n}{T}\right)^{1/2}\Psi(\chi_{[t_{j-1},t_j]})\cdot\Psi(\chi_{[t_{j-1},t_j]})\right\|_2$$

$$=\left\|\left(\frac{n}{T}\right)^{1/2}\cdot T\cdot I\right\|_2=(nT)^{1/2}\to\infty \qquad \text{as } n\to\infty$$

so that the semivariation of $d\Psi$ with respect to $L^2(\mathscr{C}_{\mathbf{T}})$ is infinite. Note that $x_j\in\mathscr{C}_{t_j}$ (rather than $\mathscr{C}_{t_{j-1}}$) so the sum we considered is not a *belated sum* We can also provide an example of how 3.3 fails by using (Ψ_t). Let us consider the function $s\to{}^h\Psi_s$. Now (4.21) indicates that $h\colon\mathbf{T}\to\mathscr{C}_{\mathbf{T}}$ is $\|\cdot\|_2$-continuous. Taking $f(s)\equiv I$ and $z\equiv 1$ in 4.22 shows that $\|\cdot\|^b_{d\Psi}=\sqrt{\lambda(\cdot)}$, where λ is Lebesgue measure. Consider h on $[0,1]$. Using uniform continuity we can show that the simple *functions* $k_n(s)=\sum_{i=0}^n h(i/n)\,\chi_{[i/n,(i+1)/n]}(s)$ and $r_n(s)=\sum_{i=0}^n h((i+1)/2)\,\chi_{[i/n,(i+1)/n]}(s)$ converge to h μ^b-a.u. on $[0, 1]$ and hence in measure. Furthermore (k_n) and (r_n) satisfy 3.1(a, b) because h is $\|\cdot\|_2$-uniformly continuous and $\|\cdot\|^b_{d\Psi} = \sqrt{\lambda}$. Observe that

$$\int_0^1 r_n(s)\,d\Psi_s - \int_0^1 k_n(s)\,d\Psi_s = \sum_{i=0}^n \left(h\left(\frac{i+1}{n}\right) - h\left(\frac{i}{n}\right)\right)\left(\Psi\left(\frac{i+1}{n}\right) - \Psi\left(\frac{i}{n}\right)\right)$$

but $h(s)$ is just Ψ_s so the sum is

$$\sum_{i=0}^n \left(\Psi\left(\frac{i+1}{n}\right) - \Psi\left(\frac{i}{n}\right)\right)^2 = \sum_{i=0}^n \left(\int \chi_{[i/n,\,(i+1)/n]}\right)^2 \cdot I = 1 \cdot I, \text{ by (4.21)}.$$

Clearly $\int_0^1 k_n\,d\Psi$ and $\int_0^1 r_n\,d\Psi$ cannot converge to the same element of $L^2(\mathscr{C}_T)$. This demonstrates that we cannot extend our definition of integrability to functions "defined by" a sequence of simple functions (rather than simple processes) because the integral would not be well defined as the limit of integrals of simple things.

4.3. *Integration under Standing Hypothesis*

(i) In [5] there is a discussion of processes that satisfy the "standing hypotheses" of McShane [11] (and below).

One can integrate suitable processes with respect to processes that satisfy the standing hypothesis. It turns our that this is a further example of the belated integral. The context is that of 4.1 with 4.1(iii) augmented with the condition $(\bigcup_{\beta<\alpha} \mathscr{A}_\alpha)'' = \mathscr{A}_\alpha$; i.e., the nest of the von Neumann algebras is continuous.

4.31. Definition. Let $2 \leqslant p \leqslant \infty$. An L^2 process (x_α) is said to satisfy the standing hypotheses (p) if $\exists k \in (0, \infty)$ $\forall s \in [0, T]$ and $\forall t \in [0, T]$ with $0 \leqslant t - s < 1$ we have

$$\|M_s(x_t - x_s)\|_p \leqslant K(t-s)$$
$$\|M_s(|x_t - x_s|^2)\|_{p/2} \leqslant K(t-s).$$

With such an L^2 process one defines a finitely additive set function on the field $\mathscr{G}$ comprising sets that are finite disjoint unions of left closed right open subintervals of $[0, T)$ by setting

$$dx([s, t)) = x_t - x_s, \qquad 0 \leqslant s \leqslant t.$$

Recall that $p \geqslant 2$. Let q be such that $1/p + 1/q = \frac{1}{2}$. We note immediately that there is a bilinear mulitplication between elements of L^p and elements of L^q satisfying $\|xy\|_2 \leqslant \|x\|_p \cdot \|y\|_q$ [7]. We let the nest (X_α) of Section 1 be $L^p(\mathscr{A}_\alpha)$ and put $Y = L^q(\mathscr{A}_T)$ (with dx corresponding to μ) and

$Z = L^2(\mathscr{A}_T)$. The conclusion of a sequence of Lemmas in Sections 1 and 2 of [5] is the following.

4.32. THEOREM (Corollary 2.4 of [5]). *Let h: $[0, T] \to L^q(\mathscr{A}_T)$ be a $\mathscr{G}$ simple process. Then*

$$\left\| \int_0^T dx_s h(s) \right\|_2^2 \leqslant C^2 \int_0^T \|h(s)\|_q^2 \, ds, \tag{*}$$

where $C = 2K\mathbf{T}^{1/2} + K^{1/2}$.

Bearing in mind the definition of $\int_0^T dx_s h(s)$, this result shows that dx has finite left-belated semivariation with respect to $(L_p(\mathscr{A}_\alpha))$ on $[0, T]$. Accordingly we can form the ${}^b dx$-integral of $L^q(\mathscr{A}_T)$-valued processes given by the "left analogue" of 3.1. The left integral under standing hypothesis (p) is extended to the set of processes in $L^2([0, T], ds, L^q(\mathscr{A}_T))$. The relation $(*)$ in 4.32 shows that ${}^b\|E\|_{dx} \leqslant C\sqrt{\lambda(E)}$, where $E \in \mathscr{G}$ and λ is Lebesgue measure. It is a routine matter to show that the left-belated and the (left) integral under standing hypothesis (p) agree and that the class of ${}^b dx$-integrable functions contains the processes in $L^2([0, T], ds, L^q(\mathscr{A}_T))$.

(ii) A version of McShane's integral is employed in [9]. With some restrictions this version of McShane's integral is another example of the belated integral. (In fact the integral of Definition B(III), Section 1, of [9] is called a belated integral by the author of [9].)

Let (Ω, F, μ) be a probability space. We consider an increasing filtration of σ-algebras (F_t), $t \in [a, b] \subseteq \mathbb{R}^+$: $F_{t_1} \subseteq F_{t_2} \subseteq F$ if $t_1 \leqslant t_2$. Let $G_1, ..., G_q$ and H be separable complex Hilbert spaces and denote by $\mathscr{L}^0(\Omega, F_{(t)}, \mu, G_i)$ the linear space of maps f: $\Omega \to G_i$-measurable with respect to $F_{(t)}$. Let z^i, $1 \leqslant i \leqslant q$, denote maps z^i: $[a, b] \to \mathscr{L}^0(\Omega, F, \mu, G_i)$ and suppose B: $[a, b] \times \Omega \to \mathrm{L}((G_i)\colon H)$. Here $\mathrm{L}((G_i)\colon H)$ denotes the set of continuous q-linear maps from $X_{i=1}^q G_i$ into H. In what follows vector-valued functions occur. We use $|\cdot|$ to denote the norm in a particular vector space (which should be clear from the context). Vector-valued functions, f, will be Bochner integrable [8] and we write, for $r \geqslant 1$

$$\|f\|_r = \left(\int_\Omega |f|^r d\mu \right)^{1/r}.$$

The following conditions are required of our processes.

Condition $A(r)$. (i) For $1 \leqslant i \leqslant q$ and $t \in [a, b]$, $z^i(t) \in \mathscr{L}^0(\Omega, F_t, \mu, G_i)$; i.e., z^i is adapted to (F_t).

(ii) For $1 \leqslant i \leqslant q$, $\exists K > 0$ $\exists \delta > 0$: $a \leqslant s < t \leqslant b$ and $t - s < \delta$ then for μ-almost every $\omega \in \Omega$

$$|M_s(z^i(t) - z^i(s))| \leqslant K(t - s)$$

and for $r \in \mathbb{N}^+$

$$M_s(|z^i(t) - z^i(s)|^{2p}) \leqslant K(t-s), \qquad p = 1, 2, \ldots, r.$$

Here $M_s(\cdot)$ denotes the conditional expectation operator

$$M_s \colon L^1(\Omega, F, \mu, G_i) \to L^1(\Omega, F_s, \mu, G_i)$$

and $|\cdot|$ the norm in G_i (we have omitted the dependence on G_i).

Condition $B(p)$. For $t \in [a, b]$, $B_t \overset{\text{def}}{=} B(t, \cdot)$ is an element of $\mathscr{L}^{2p}(\Omega, F_t, \mu, \mathbf{L}((G_i); H))$ and the map $t \to \|B_t\|_{2p}$ is continuous.

Now let $a \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n+1} = b$ and $\Delta t_j = t_{j+1} - t_j$, $\Delta z_j^i = z^i(t_{j+1}) - z^i(t_j)$. Suppose $B_{t_j} \in \mathscr{L}^2(\Omega, F_t, \mu, \mathbf{L}((G_i; H))$.

4.33. Lemma (III, Sect. 2, Corollary 2, of [9]). *If z^i, $1 \leqslant i \leqslant q$, satisfy $A(q)$ with constants K and δ and* $\max_j(t_{j+1} - t_j) < \delta$ *then*

$$\left\| \sum_{j=1}^{n} B_{t_j}(\Delta z_j^1, \Delta z_j^2, \ldots, \Delta z_j^n) \right\|_2 \leqslant \beta \left(\sum_{j=1}^{n} \|B_{t_j}\|_2^2 \Delta t_j \right)^{1/2},$$

where $\beta = 2K(b-a)^{1/2} + K^{1/2}$.

This is another result about belated semivariation of a vector measure. To see this put $X_{(t)} = \mathscr{L}^2(\Omega, F_{(t)}, \mu, \mathbf{L}((G_i; H))$ and let $Y = X_{i=1}^q H_i$ where $H_i = \{z; z\colon [a, b] \to \mathscr{L}^0(\Omega, F, \mu, G_i)$ and z is adapted to $(F_t)\}$. Let z^i, $1 \leqslant i \leqslant q$, satisfy $A(q)$ and define a measure on subintervals of $[a, b]$ by

$$dz([s, t]) = (z^1(t) - z^1(s), z^2(t) - z^2(s), \ldots).$$

Let the multiplication on $X \times Y$ be defined by $B \cdot z = B(z)$ so that $B(z) \in \mathscr{L}^2(\Omega, F, \mu, H) = Z$ (of Section 1). The lemma above shows that dz has finite-belated semivariation with respect to (X_t) on $[a, b]$. We note immediately that if (B_t) satisfies $B(1)$ then it is dz^b-integrable. Theorem 3 of III, Section 3 of [9] indicates that the belated integral of (B_t) coincides with that defined in [9].

Remark. In the lemma it was stipulated that the mesh of the partition should be less than δ. This does not affect the issue, for any simple process may be written as $\sum_i B_{t_i} \chi_{[t_i, t_{i+1})}$ with $\max_i |t_{i+1} - t_i| < \delta$.

4.4. *Quasi-free Integrals for the CAR*

We take our notation from [3], in particular Sections 1 and 2. So let A (resp. A_t) denote the CAR C^*-algebra over $L^2(\mathbb{R}^+)$ (resp. $L^2([0, t])$). Let $\mathscr{H}$ (resp. $\mathscr{H}_t$) denote the Hilbert space given by $A\Omega_R$ (resp. $A_t\Omega_R$). Here Ω_R is the GNS vector corresponding to the quasi-free state on A given by

$w(b^*(u)\,b(v)) = \langle v, Ru\rangle$, $u, v \in L^2(\mathbb{R}^+)$, and $R \in B_1(L^2(\mathbb{R}^+))^+$ is equivalent to multiplication by some function $\rho(\cdot)$ (see Section 2 of [3]).

We can consider A_t as a subalgebra of A and $\mathscr{H}_t$ as a closed subspace of H. Let $X_t = \alpha_1 B^*(u_t) + \alpha_2 B(u_t)$, with $u_t = u\chi_{[0,t)}$, $u \in L^2_{\text{loc}}(\mathbb{R}^+)$, be the A-valued martingale of Section 3 Definition 3.1, of [3]. Define

$$V([s,t]) = X_t - X_s \in A, \qquad V(\{s\}) = 0; \qquad 0 \leqslant s \leqslant t.$$

Consider the pairing $\mathscr{H} \times A \to \mathscr{H}$ given by $(\xi, a) \to a\xi$, and let $\{\mathscr{H}_t\}_{t\in\mathbb{R}}$ define the filtration of $\mathscr{H}$.

4.41. LEMMA. *For each bounded $I \in \mathscr{F}$, V has finite belated semivariation over I. In fact $\|I\|^b_V = \mu(I)^{1/2}$, where μ is the measure $d\mu(s) = \{|\alpha_1|^2(1-\rho(s)) + |\alpha_2|^2\rho(s)\}\,|u(s)|^2\,ds$, where $0 \leqslant \rho(s) \leqslant 1$, determines the quasi-free state on the CAR algebra.*

Proof. Let $I = \bigcup_{j=1}^N I_j$ be a partition of I into disjoint intervals I_j. Let $t_j = \inf I_j$, $t'_j = \sup I_j$. Then for any $\xi_j \in \mathscr{H}_{t_j}$ with $|\xi_j| \leqslant 1$ we have

$$\begin{aligned}\Big|\sum_j V(I_j)\,\xi_j\Big|^2 &= \Big|\sum_j (X_{t'_j} - X_{t_j})\,\xi_j\Big|^2 \\ &= \sum_j \int_{t_j}^{t'_j} |\xi_j|^2\,d\mu \qquad \text{as in [3]} \\ &\leqslant \sum_j \int_{I_j} d\mu = \mu(I).\end{aligned}$$

Taking $\xi_j = \Omega_R$, we have

$$\Big|\sum_j V(I_j)\,\Omega_R\Big|^2 = \int_I d\mu = \mu(I).$$

Hence $\|I\|^b_V = \mu(I)^{1/2}$. Q.E.D.

So μ is a control measure for V and the results of Section 2 apply. Accordingly for any bounded interval $I \subseteq \mathbb{R}^+$ there is a class of V^{b_σ}-integrable functions on I. Now as $\|E\|^b_V = \mu(E)^{1/2}$ for $E \in \mathscr{F}$ it follows that for any $A \subseteq I$ that $\|A\|^{b_\sigma}_V$ is small if and only if its outer μ-measure is small. So a V^{b_σ}-measurable process is a μ-measurable process and conversely. By 2.15 $\|\cdot\|^{b_\sigma}_V$ is countably subadditive on the power set of I. It follows from 2.7 that a sequence converging in V^{b_σ}-measure has a subsequence converging V^{b_σ}-a.e. We shall use this fact in our next result.

4.42. THEOREM. *Let $I \subset \mathbb{R}$ be a finite interval. Then $f\colon I \to \mathscr{H}$ is V^{b_σ}-integrable on I if and only if f is a process in $\mathscr{L}^2(I, d\mu; \mathscr{H})$* (i.e., *if and only if $f \in \mathscr{L}^2(I, d\mu; \mathscr{H})$ and $f(s) \in \mathscr{H}_s$ μ.a.e. on I*).

Proof. Suppose that f is V^{b_σ}-integrable on I. Then there is a sequence (g_n) of simple $\mathscr{H}$-valued processes on I such that $g_n \to f$ on I in V^{b_σ}-measure, and $\int_I g_n\, dV^{b_\sigma}$ converges in $\mathscr{H}$ to $\int_I f\, dV^{b_\sigma}$. Hence $(\int_I g_n\, dV^{b_\sigma})$ is a Cauchy sequence in $\mathscr{H}$. But $\int_I g_n\, dV^{b_\sigma} = \int dV g_n = \int dX g_n$, the stochastic integral as given in [3]. By the isometry property [3], we have

$$\left|\int_I (g_n - g_m)\, dV^{b_\sigma}\right|^2 = \left|\int_I dV(g_n - g_m)\right|^2$$

$$= \int_I |g_n - g_m|^2\, d\mu.$$

If follows that (g_n) is a Cauchy sequence in $\mathscr{L}^2(I, d\mu; \mathscr{H})$. Hence there exists $F \in \mathscr{L}^2(I)\, d\mu; \mathscr{H})$ such that $F(s) \in \mathscr{H}_s$ μ-a.e. on I and $g_n \to F$ in $\mathscr{L}^2(I, d\mu; \mathscr{H})$. Thus there is a subsequence (g_{n_k}) with $g_{n_k} \to F$ μ-a.e. on I.

Now $g_{n_k} \to f$ in V^{b_σ}-measure, and hence there is a subsequence $(g_{n_{k_l}})$ such that $g_{n_{k_l}} \to f$ V^{b_σ}-a.e. on I (by Proposition 3.7). By the lemma, it follows that $F = f$ V^{b_σ}-a.e.

Conversely, suppose that $f \in \mathscr{L}^2(I, d\mu; \mathscr{H})$ with $f(s) \in \mathscr{H}_s$ μ-a.e. on I. Then we know [3] that there is a sequence (g_n) of simple processes on I such that $g_n \to f$ in $\mathscr{L}^2(I, d\mu; \mathscr{H})$. Hence $g_n \to f$ in μ-measure. By the lemma, it follows that $g_n \to f$ in V^{b_σ}-measure. It remains to show that (g_n) is uniformly absolutely continuous with respect to V^b. By the isometry property, for $E \subseteq I$, $E \in \mathscr{F}$, $|\int_E g_n\, dV^{b_\sigma}| = \{\int_E |g_n|^2\, d\mu\}^{1/2}$. Hence for given $\varepsilon > 0$

$$\left\{\int_E |g_n|^2\, d\mu\right\}^{1/2} \leqslant \left\{\int_I |g_n - f|^2\, d\mu\right\}^{1/2} + \left\{\int_E |f|^2\, d\mu\right\}^{1/2}$$

$$< \varepsilon + \left\{\int_E |f|^2\, d\mu\right\}^{1/2} \qquad \text{for all sufficiently large } n,$$

$$< 2\varepsilon \qquad \text{for sufficiently small } \mu(E).$$

Hence $\{\int_E |g_n|^2\, d\mu\}^{1/2} \to 0$ as $\mu(E) \to 0$ uniformly in n. By the lemma, we deduce that $\int_E g_n\, dV^{b_\sigma} \to 0$ as $\|E\|_V^{b_\sigma} \to 0$ uniformly in n. Thus f is V^{b_σ}-integrable on I.

4.43. COROLLARY. *For any process $f \in \mathscr{L}^2(I, d\mu; \mathscr{H})$, we have $\int_I dXf = \int_I f\, dV^{b_\sigma}$, where $\int_I dXf$ is the left-stochastic integral of (the class determined by) f as constructed in* [3].

Proof. We have seen that a process f is V^{b_σ}-integrable over I if and only if $f \in \mathscr{L}^2(I, d\mu; \mathscr{H})$. Using Theorem 3.3 and the definition of the left-stochastic integral the result follows. Q.E.D.

4.44. *Remark.* The right integral can be obtained by considering the pairing $\mathscr{H} \times A \to \mathscr{H}$, $(\xi, a) \to -\Gamma a\xi$, where Γ implements the parity automorphism β [3].

Suppose that the quasi-free state has no Fock part. Let $\mathscr{H}_{+1}$, etc., be as in [3].

We consider the A-valued measure V as before, but now we wish to consider $\mathscr{H}_{+1}$-valued integrands, and $\mathscr{H}_{+1}$-valued integrals.

There is a pairing $A\Omega_R \times A \to \mathscr{H}_{+1}$ given by $(x\Omega_R, a) \to ax\Omega_R$, where $A\Omega_R$ is considered as a (dense) subset of $\mathscr{H}_{+1}$. We will consider the filtration $\{A_t\Omega_R\}_{t\in\mathbb{R}^+}$ of $\mathscr{H}_{+1}$.

4.45. PROPOSTITION. *Let $E \in \mathscr{F}$. Then*

$$\frac{1}{\sqrt{2}}(|\alpha_1|^2 + |\alpha_2|^2)^{1/2}\,\gamma(E)^{1/2} \leqslant \|E\|_V^b \leqslant (|\alpha_1|^2 + |\alpha_2|^2)^{1/2}\,\gamma(E)^{1/2},$$

where $d_\gamma = |u(s)|^2\,ds$.

Proof. Let $E = \bigcup_j I_j$ be a partition of E into disjoint intervals and let $t_j = \inf I_j$, $t'_j = \sup I_j$. Then, for $x_j \in A_{t_j}$ with $\|x_j\Omega_R\|_{+1} \leqslant 1$, we have

$$\begin{aligned}\Big\|\sum_j V(I_j)\,x_j\Omega_R\Big\|_{+1}^2 &= \Big\|\sum_j (X_{t'_j} - X_{t_j})\,x_j\Omega_R\Big\|_{+1}^2 \\ &= \int_E \|x(s)\,\Omega\|_{\rho,\Delta}^2\,d_\gamma(s)\end{aligned}$$

by the isometry property, Theorem 4.1 of [3], where $x(s) = \sum_j \chi_{I_j}(s)\,x_j\Omega_R$ and $d_\gamma(s) = |u(s)|^2\,ds \leqslant \int_E (\lambda_{\alpha_1,\alpha_2} + \lambda_{\alpha_2,\alpha_1})\,d\gamma = (|\alpha_1|^2 + |\alpha_2|^2)\,\gamma(E)$. Taking $x_j = (1/\sqrt{2}).\ I$ gives

$$\|E\|_V^b \geqslant \frac{1}{\sqrt{2}}(|\alpha_1|^2 + |\alpha_2|^2)^{1/2}\,\gamma(E)^{1/2}. \qquad \text{Q.E.D.}$$

4.46. THEOREM. *$f: I \to \mathscr{H}_{+1}$ is $V^{b\sigma}$-integrable over a finite interval I if and only if $f \in \mathfrak{H}_{\rho,\Delta}(I)$, here $\mathfrak{H}_{\rho,\Delta}(I)$ is defined as in Definition* 4.2 *of* [3]. *Moreover the stochastic integral $\int_I dXf$ is equal to the belated integral $\int f\,dV^b$. (Strictly speaking we should consider the class of f in $\mathfrak{H}_{\rho,\Delta}(I)$.)*

Proof. Follow the argument of 4.42. Q.E.D.

4.5. *Quasi-free Integrals for the CCR*

Let $\tau \in L^\infty_{\text{loc}}(\mathbb{R}^+)$, $\tau \geqslant 0$, and let Ω be the cyclic vector corresponding to the CCR quasi-free state given by

$$\omega(a^*(f)\,a(g)) = \int_0^\infty \tau(s)\,f(s)\,\overline{g(s)}\,ds$$

for f, g in the domain of the multiplication by $\tau^{1/2}$ on $L^2(\mathbb{R}^+)$. Let $\mathscr{P}$ be the unital polynomial algebra generated by the $a^*(f)$, $a(g)$ in the cyclic representation given by Ω and where f, g run over the domain of $\tau^{1/2}$, and $\mathscr{P}_t$ that subalgebra of $\mathscr{P}$ where the f and g are restricted to have support in $[0, t]$. For details see [3]. Put $X = \mathscr{P}\Omega$, $X_t = \mathscr{P}_t\Omega$, $Y = \mathscr{P}$ and $Z = \mathscr{P}\Omega$. We observe that $\mathscr{P}\Omega$ is normed but not complete. The pairing is $x \in X$, $y \in Y$, $(x, y) \to yx \in \mathscr{P}\Omega$. Define V by

$$V([s, t]) = Y_t - Y_s \in \mathscr{P}, \qquad V(\{s\}) = 0,$$

where $Y_t = \alpha_1 A_t^*(u) + \alpha_2 A_t(u)$, where $A_t^*(u) = a^*(\chi_{[0, t]} \cdot u)$, $A_t = a(\chi_{[0, t]} \cdot u)$, and $u \in L^2_{\text{loc}}(\mathbb{R}^+)$. One can establish the following analogues of the results in 4.41–4.46 with similar proofs. For the notation and further details we refer to [3].

4.51. PROPOSITION. *Let $E \in \mathscr{F}$. Then $\|E\|_V^b = \nu(E)^{1/2}$, where*

$$d\nu = \{|\alpha_1|^2(1 + \tau(s)) + |\alpha_2|^2 \tau(s)\} \, |u(s)|^2 \, ds.$$

Proof. Just use the isometry property Theorem 5.8 of [3]. Q.E.D.

4.52. THEOREM. *f is V^{b_σ}-integrable on a finite interval I iff f is a process in $\mathscr{L}^2(I, d\nu; \mathscr{H})$. The belated integral is equal to the stochastic integral.*

Now take $X = \mathscr{P}\Omega \subseteq \mathscr{H}_{+1}$, $Y = \mathscr{P}$, $Z = \mathscr{P}\Omega \subseteq \mathscr{H}_{+1}$, where $\tau > 0$ and $\mathscr{H}_{+1}$ is the Sobolev space given by the modular operator for Ω. Then we have

4.53. LEMMA. *For $E \in \mathscr{F}$*

$$\frac{1}{\sqrt{2}}(|\alpha_1|^2 + |\alpha_2|^2)^{1/2} \gamma'(E)^{1/2} \leqslant \|E\|_V^b \leqslant (|\alpha_1|^2 + |\alpha_2|^2)^{1/2} \gamma'(E)^{1/2},$$

where $\gamma'(E) = \int_E (1 + 2\tau(s)) \, d\gamma(s)$.

Proof. Use the isometry property for $\mathscr{H}_{+1}$—Theorem 6.6 of [3].

4.54. THEOREM. *f is V^{b_σ}-integrable over a bounded interval I iff f belongs to (a class in) $\mathfrak{H}_{t,\Delta}(I)$. (See [3] for the definition of $\mathfrak{H}_{t,\Delta}(I)$.)*

4.55. *The Itô Integral with respect to Brownian Motion*

As noted in [3], a special case of the quasi-free CCR stochastic integral reduces to an Itô stochastic integral.

Indeed, take $t = 0$, $u = 1$, $\alpha_1 = \alpha_2 = 1$. Let $\phi_s = a^*(\chi_{[0, s]}) + a(\chi_{[0, s]})$. Then $\{\phi_s : s \in \mathbb{R}\}$ is a family of jointly Gaussian random variables with mean zero and $\text{cov}(\phi_s, \phi_t) = \min\{s, t\}$. Let $X = \overline{\mathscr{P}(\phi)\,\Omega} = \mathscr{H}$, $Y = \mathscr{P}(\phi)$, $z =$

$\overline{\mathscr{P}(\phi)\,\Omega} = \mathscr{H}$ with filtration $\mathscr{P}_t(\phi)\,\Omega\}_{t\in\mathbb{R}}$ of X, where $\mathscr{P}_t(\phi)$ denotes the set of polynomials in ϕ_s for $s \leqslant t$, and $\mathscr{P}(\phi) = \bigcup_t \mathscr{P}_t(\phi)$.

Let $W([s, t]) = \phi_t - \phi_s$, $W(\{s\}) = 0$, define a vector measure on $\mathscr{F}$. Let $E \in \mathscr{F}$. Then if $E = \bigcup_j I_j$ is a finite partition of E into intervals, we have, for $x_i \in \mathscr{P}_{t_i}(\phi)$, $\|x_i\Omega\| \leqslant 1$, where $t_i = \inf I_i$,

$$\left\|\sum_j x_j \Omega W(I_j)\right\|^2 = \left\|\sum_j (\phi_{t'_j} - \phi_{t_j})\, x_j\Omega\right\|^2, \qquad \text{where } t'_j = \sup I_j$$

$$= \sum_j \|x_j\Omega\|^2 \lambda(I_j) \leqslant \lambda(E), \qquad \text{where } \lambda = \text{Lebesgue measure.}$$

Taking x_j to be the identity operator gives $\|E\|^b_w = \lambda(E)^{1/2}$. Moreover, as before, f is W^{b_σ}-integrable on I if and only if f is (in a class) in $\mathfrak{R}(I, ds; \mathscr{P}(\phi))$; and the W^{b_σ}-integral is equal to the CCR stochastic integral which is equal to the Itô stochastic integral of f.

4.6. *Concluding Remarks*

We have looked at one way of generalising the construction of the belated integral and it did not work. If f is an elementary process, e.g., $f(s) = x\chi_{[t_1, t_2]}(s)$ then we require $x \in X_{t_1}$. The idea of the generalisation is to allow x to belong to some X_α for $\alpha \in [t_1, t_2]$ but not necessarily X_{t_1}. More precisely let a *choice function* be a map $\theta: \mathbb{R}^2 \to \mathbb{R}$ such that $\theta(x, y) = \theta(y, x)$ and $\theta(x, y) \in [x, y]$. If I is an interval with $a = \inf I$, $b = \sup I$ then $\theta(I) \overset{\text{def}}{=} \theta(a, b) \in I \cup \{a, b\}$. We generalise $\|\cdot\|^b$ to the θ-semivariation; that is, for $E \in \mathscr{F}$

$$\|E\|^b_{\theta,\mu} = \sup \left|\sum_i x_i \mu I_i\right|,$$

where the supremum is taken over partitions $\{I_i\}$ of E, into intervals I_i and over all choices of $x_i \in X_{\theta(I_i)}$ with $|x_i| \leqslant 1$. The definition of θ-simple functions now follows. But such functions are *not* a linear space—an essential piece of structure for the belated integral. To see this consider the following.

EXAMPLE. Let $\theta([0, 1]) = \beta$. Let $y \in X_\beta \backslash \bigcup_{\alpha<\beta} X_\alpha$; then $f(s) = y\chi_{[0,t]}(s)$ is θ-simple. Let $g(s) = x_1\chi_{[0,\beta]}(s) + x_2\chi_{[\beta,1]}(s)$ be a θ-simple process. The pointwise sum of f and g is $(x_1 + y)\,\chi_{[0,\beta]}(s) + (x_2 + y)\,\chi_{[\beta,1]}(s)$. But $x_1 + y \in X_\beta$, so if $\inf I < \theta(I) < \sup I$ for each interval I then $f + g$ is not θ-simple.

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